OLLSCOIL NA hÉIREANN GAILLIMH

NATIONAL UNIVERSITY OF IRELAND GALWAY

SUMMER EXAMINATIONS 1999

MA 387 (PROBABILITY)

THIS EXAMINATION MAY BE TAKEN ONLY BY **REPEAT** CANDIDATES

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Time allowed: *three* hours.
Attempt *five* questions.

1. (i) Explain briefly the difference between *probability* and *statistics*.
   (ii) State the Kolmogorov axioms for the probability $P(A)$ of an event A. Then deduce that
$$P(\emptyset) = 0,$$
$$P(\overline{A}) = 1 - P(A),$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
   (iii) State (without proof) a formula for $P(A_1 \cup A_2 \cup \cdots \cup A_{52})$.
   (iv) Determine the probability of at least one perfect match occurring when two shuffled packs of 52 cards are dealt simultaneously and separately to two people.

2. (i) Suppose that in a series of three Wales v. Ireland rugby matches: Wales has a 70% chance of winning their first game; if a team wins one match then its chances in the next match are increased by 10%. Determine the probability that Wales wins the series (i.e. that Wales wins two matches).
   (ii) State and prove Bayes' Theorem.
   (iii) Assume the probability is 0.90 that the jury selected to try a criminal case will arrive at the correct verdict of innocent or guilty. Further, suppose that the police force are diligent in performing their duties, and 99% of the people brought to trial are in fact guilty. Given that a jury finds a certain defendant guilty, what is the probability that she is in fact guilty?
   (iv) Tom and Mary take turns at throwing a dice. Tom throws first. The winner is the first to obtain a 6. Determine the probability that Tom wins.

3. (i) Your waiting time X at the supermarket counter has the following probability density function:
$$f_X(x) = \frac{2x}{b^2} \quad \text{for } 0 < x < b, \qquad f_X(x) = 0 \quad \text{otherwise.}$$
   Evaluate the mean, the standard deviation, the median, and the cumulative distribution function of X.
   (ii) State and prove Chebyshev's inequality.

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4. (i) Let X denote the number of cards dealt from a shuffled pack of 52 before the second ace appears. Determine the mean and variance of X.

   (ii) State and prove a formula for the variance $var(X+Y)$ of the sum of two random variables X, Y given in terms of the variances and covariance of X and Y.

   (iii) You are asked to carry out an opinion poll to estimate how many people will vote *yes* in the forthcoming referendum on the Amsterdam Treaty. Use the Central Limit Theorem (or Chebyshev's Theorem) to determine a sample size big enough to be at least 90% certain that your estimate is within 5% of the actual number. List some of the simplifying assumptions that you have made in your calculations.

5. (i) The conditional probability density function for X, given $Y = y$, is $f_{X|Y}(x|y) = \frac{1}{y^2}$ for $0 < x < y^2$, while the marginal probability density function for Y is $f_y(y) = 4y^3$ for $0 < y < 1$. Determine the marginal density function for X. Are X and Y independent?

   (ii) Suppose a rectangle is constructed in the xy-plane with base length x and height y, where x is the observed value of a random variable X that is uniform on the interval $(1, 2)$, while y is the observed value of a random variable Y that, given $X = x$, is uniform on the interval $(0, x)$. Determine the expected area of the rectangle.

6. (i) Telephone calls arrive at a police station at an average rate of λ per hour, and follow a Poisson distribution. You begin to monitor the calls at time $t = 0$ and let $t = Y$ denote the time of the first call. Explain why the cumulative distribution function of Y is $F_Y(t) = 1 - e^{-\lambda t}$ for $t \geq 0$, and $F_Y(t) = 0$ otherwise. What is the probability that the first call will arrive after t=2 hours, given that no call arrives in the first hour and that $\lambda = 2$?

   (ii) The moment generating function for a discrete random variable Y is $m_Y(t) = e^{t(5+t)}$. Determine the mean, the variance, and the probability density function of Y.

7. (i) Let Z be a standard normal random variable. Prove that the moment generating function of Z is $m_Z(t) = e^{\frac{t^2}{2}}$.

   (ii) Let $X_1, X_2, \cdots$ be a sequence of independent random variables, each with mean 0 and variance 1, and moment generating function $m_X(t)$. Let $Y = \frac{X_1+\cdots+X_n}{\sqrt{n}}$. Prove that

$$\lim_{n\to\infty} m_Y(t) = e^{\frac{t^2}{2}}.$$

   (iii) State and prove the Central Limit Theorem.

   (iv) Prove that for large n, a binomial random variable X with parameters n, p is approximately normally distributed.

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(i) The owner of a small firm is forced to lay off 3 of her 20 employees. Eleven of the employees are female. If the three to be laid off are chosen at random, what is the probability that most of those laid off are male?

(ii) A sequence of independent Bernoulli trials is carried out, each with probability p of success. Let N be the random trial number of the first success. Derive formulae for the mean and cumulative distribution function of N.

(iii) Show, by means of an example, that $Cov(X, Y) = 0$ does not imply that two jointly distributed random variables X, Y are independent.

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