GX1611

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

B.Sc. (Part II) EXAMINATION
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS

MA481 = MA490 [MEASURE THEORY] — & — MA482 [FUNCTIONAL ANALYSIS]

Professor D.H. Armitage
Professor T.C. Hurley
Dr. D. O'Regan
Dr. R. Ryan.

Time allowed: **Three** hours.
Full marks for five questions.

SECTION A — *MEASURE THEORY*

A1. (a) Let $\mathcal{M}$ be a σ-algebra of subsets of X and suppose $\{A_n\}$ is a sequence of sets of $\mathcal{M}$. Define $\limsup A_n$.
If B_n is a sequence of sets of $\mathcal{M}$ show

$$\limsup(A_n \cap B_n) \subseteq (\limsup A_n) \cap (\limsup B_n).$$

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space with

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$$

Show

$$\mu\left(\bigcup_1^\infty E_k\right) = \lim_{n\to\infty} \mu(E_n).$$

(c) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{A_n\}$ a sequence of measurable sets. Show

$$\mu(\liminf A_n) \le \liminf \mu(A_n).$$

p.t.o.

A2. Let $\mathcal{L}$ denote the class of Lebesgue measurable set in $\mathbf{R}^n$.

(a) If $\{E_k\}_k^\infty$ is a sequence of subsets of $\mathcal{L}$, show

$$\bigcup_{k=1}^{\infty} E_k \in \mathcal{L}.$$

(b) Let $a > 0$ and $E \subseteq \mathbf{R}^n$.

(i) Show $| aE |_o = a^n | E |_o$;

(ii) If $E \in \mathcal{L}$ show $aE \in \mathcal{L}$.

(c) Let $E \subseteq \mathbf{R}^n$. Show the following are equivalent:

(i) $E \in \mathcal{L}$;

(ii) There exists a G_δ set $H \supseteq E$ with $| H \backslash E |_o = 0$;

(iii) for each $\epsilon > 0$ there exists a closed set $F \subseteq E$ with $| E \backslash F |_o < \epsilon_0$;

(iv) There exists a F_σ set $K \subseteq E$ with $| E \backslash K |_o = 0$;

(v) There exists a Borel set $B \in \mathcal{B}(\mathbf{R}^n)$ and a Lebesgue measurable set A with $| A |_o = 0$ and $E = B \cup A$.

A3. Let $(X, \mathcal{M}, \mu)$ be a measure space and f an extended real valued function defined on X.

(a) Let $A = f^{-1}(\{\infty\})$ and $B = f^{-1}(\{-\infty\})$ and $g : X \to \mathbf{R}$ be defined by

$$g(x) = \begin{cases} f(x), & x \in X \backslash (A \cup B) \\ 0, & x \in A \cup B. \end{cases}$$

If f is measurable show g is measurable.

(b) Suppose f is measurable. Show $f^{-1}(B) \in \mathcal{M}$ for every Borel set $B \subseteq \mathbf{R}^n$.

(c) Let f be nonnegative and measurable. Show that there is a nondecreasing sequence of measurable simple real valued functions S_n with

$$\lim_{n \to \infty} S_n(x) = f(x) \quad \text{for} \quad x \in X.$$

A4. (a) *State* the Lebesgue dominated convergence theorem.
Use it to show

$$\lim_{n \to \infty} \int_0^2 e^{-x} \cos\left(\frac{x}{n^2}\right) \, dx = 1 - e^{-2}.$$

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space, I an interval in $\mathbf{R}$, $f : X \times I \to \mathbf{R}$ and $t_0 \in I$. Suppose the following conditions held:

(i) $f(\,.\,, t)$ is μ-integrable for every $t \in I$;

(ii) $f(x, \,.\,)$ is continuous at t_0 for each $x \in X$;

(iii) there exists a μ-integrable function $g : X \to \mathbf{R}$ such that $| f(x,t) | \leq g(x)$ for all $x \in X$ and for all $t \in I$ in some neighborhood of t_0.

Show

$$F(t) = \int_X f(x,t) \, d\mu(x)$$

is continuous at $t_0 \in I$.

(c) Let

$$F(t) = \int_0^\infty e^{-2xt} \frac{\sin x}{x} \, dx \quad \text{for} \quad t > 0.$$

Show F is continuous on $\{t : t > 0\}$.

SECTION B — *FUNCTIONAL ANALYSIS*

B1. (a) What is a *convex set*? Show that the closed unit ball of a normed space is convex. Show that the set

$$B = \{x \in \mathbf{R}^2 : |x_1|^{1/2} + |x_2|^{1/2} \leq 1\}$$

is not convex and sketch this set.

(b) State the Hölder and Minkowski inequalities and prove *one* of them.

(c) What is meant by saying that two norms are *equivalent*? Give the definition of the norms $\|.\|_1$ and $\|.\|_2$ on $\mathbf{R}^n$ and show that they are equivalent. Give an example, with proof, of two norms on the space $C[0,1]$ that are not equivalent.

B2. (a) Give the definition of the *norm*, $\|T\|$, of a bounded linear mapping $T: X \to Y$ between normed spaces. Let X be $\mathbf{R}^2$ with the ℓ_∞-norm, let Y be $\mathbf{R}^2$ with the ℓ_2-norm and let $T(x) = (2x_1, x_1 - 3x_2)$. Find the norm of T.

(b) Let $\mathcal{L}(X;Y)$ be the space of bounded linear mappings from X into Y with the norm as defined in part (a). Show that $\mathcal{L}(X;Y)$ is complete if Y is complete.

(c) Let $T: X \to Y$ be a bounded invertible linear mapping with inverse T^{-1}. How is $\|T^{-1}\|$ related to $\|T\|$?

B3. (a) What is the *dual space*, X^*, of a normed space X? Show that the dual space of ℓ_p is isometrically isomorphic to ℓ_q, where $1 < p < \infty$ and $1/p + 1/q = 1$.
(Note: you may assume that the unit vectors e_n form a Schauder basis for ℓ_p, i.e., the series $\sum x_n e_n$ converges to x for every $x = (x_n) \in \ell_p$.)

(b) Answer **either** (i) **or** (ii):

(i) State the Hahn-Banach Theorem. Use it to show that every normed space can be considered as a subspace of its second dual space.

(ii) Let Y be a subspace of a real normed space X and let $\varphi \in Y^*$. Let $z \in X$, $z \notin Y$, and let Z be the subspace of X that is spanned by Y and z. Show that there is a bounded linear functional ψ on Z such that

$$\psi(y) = \varphi(y) \quad \forall y \in Y \qquad \text{and} \qquad \|\psi\| = \|\varphi\|.$$

B4. (a) Let H be an inner product space. Prove the Parallelogram Law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for every $x, y \in H$. Hence or otherwise show that the space $C[0,1]$ with the norm $\|f\| = \sup\{|f(t)| : 0 \leq t \leq 1\}$ is not an inner product space.

(b) Let H be a Hilbert space and let F be a closed subspace of H. Give the definition of the *orthogonal complement*, $F^\perp$, and show that it is a closed subspace of H. Show that for every $x \in H$ there exist unique $x_1 \in F$ and $x_2 \in F^\perp$ such that $x = x_1 + x_2$.