GX1612

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 1999

B.Sc. (Part II) EXAMINATION
OCCASIONAL STUDENTS
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS

MA482 — *FUNCTIONAL ANALYSIS*

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Time allowed: **Two** hours.
Full marks for three questions.

1. (a) What is a *convex set*? Show that the closed unit ball of a normed space is convex. Show that the set

$$B = \{x \in \mathbf{R}^2 : |x_1|^{1/2} + |x_2|^{1/2} \leq 1\}$$

is not convex and sketch this set.

(b) State the Hölder and Minkowski inequalities and prove *one* of them.

(c) What is meant by saying that two norms are *equivalent*? Give the definition of the norms $||.||_1$ and $||.||_2$ on $\mathbf{R}^n$ and show that they are equivalent. Give an example, with proof, of two norms on the space $C[0,1]$ that are not equivalent.

2. (a) Give the definition of the *norm*, $||T||$, of a bounded linear mapping $T: X \to Y$ between normed spaces. Let X be $\mathbf{R}^2$ with the ℓ_∞-norm, let Y be $\mathbf{R}^2$ with the ℓ_2-norm and let $T(x) = (2x_1, x_1 - 3x_2)$. Find the norm of T.

(b) Let $\mathcal{L}(X;Y)$ be the space of bounded linear mappings from X into Y with the norm as defined in part (a). Show that $\mathcal{L}(X;Y)$ is complete if Y is complete.

(c) Let $T: X \to Y$ be a bounded invertible linear mapping with inverse T^{-1}. How is $||T^{-1}||$ related to $||T||$?

3. (a) What is the *dual space*, X^*, of a normed space X? Show that the dual space of ℓ_p is isometrically isomorphic to ℓ_q, where $1 < p < \infty$ and $1/p + 1/q = 1$.
(Note: you may assume that the unit vectors e_n form a Schauder basis for ℓ_p, i.e., the series $\sum x_n e_n$ converges to x for every $x = (x_n) \in \ell_p$.)
(b) Answer **either** (i) **or** (ii):
   (i) State the Hahn-Banach Theorem. Use it to show that every normed space can be considered as a subspace of its second dual space.
   (ii) Let Y be a subspace of a real normed space X and let $\varphi \in Y^*$. Let $z \in X$, $z \notin Y$, and let Z be the subspace of X that is spanned by Y and z. Show that there is a bounded linear functional ψ on Z such that

$$\psi(y) = \varphi(y) \quad \forall y \in Y \qquad \text{and} \qquad \|\psi\| = \|\varphi\|.$$

4. (a) Let H be an inner product space. Prove the Parallelogram Law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for every $x, y \in H$. Hence or otherwise show that the space $C[0,1]$ with the norm $\|f\| = \sup\{|f(t)| : 0 \le t \le 1\}$ is not an inner product space.
(b) Let H be a Hilbert space and let F be a closed subspace of H. Give the definition of the *orthogonal complement*, $F^\perp$, and show that it is a closed subspace of H. Show that for every $x \in H$ there exist unique $x_1 \in F$ and $x_2 \in F^\perp$ such that $x = x_1 + x_2$.