GX 1675

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2000

B.Sc (Honours) Degree

Mathematical Science/Computing

MP328 (Calculus of Variations)

Professor M.J. Sewell;
Professor J.N. Flavin;

Time allowed: *TWO* hours.
THREE questions to be attempted. Full marks are given for **TWO** and a **HALF** questions.

1. (a) Determine the minimum value of

$$\int_0^1 (y' + y)^2 dx$$

in the class of smooth functions $y = y(x)$ such that $y(0) = 0, y(1) = 1$

(b) Determine the minimum value when the condition $y(0) = 0$ is omitted.

By comparing the two minima, prove that the one arising in the second case is smaller that that arising in the first case. State why this is to be expected.

2. Consider the boundary value problem

$$[T(x)y']' - k(x)y = -w(x)$$

$$y(0) = 0, y'(\ell) = 0$$

where $T(> 0), k(> 0)$, w are smooth functions. Prove that

$$\int_0^\ell \left[\frac{(U' + w)^2}{k} + \frac{U^2}{T} \right] dx \geq \int_0^\ell wy\,dx$$

where U is an arbitrary smooth function satisfying one restriction which must be stated. When is the equality sign realized ?

By taking $U = aTy_0'$, where $y_0(x)$ denotes the solution of the boundary value problem when $k \equiv 0$ and α is a suitable constant, prove that

$$\frac{\int w^2/k\,dx \int wy_0 dx}{\int (w^2/k + wy_0)dx} \geq \int wy\,dx.$$

3. State and prove Schwarz's inequality for a linear, symmetric positive definite operator.
Prove that the operator arising in *Question* 2 is linear symmetric positive definite.
If $\bar{y}_0$ denotes the solution of the boundary value problem arising in *Question* 3 when $w(x)$ is replaced by $\bar{w}\ (x)$ and when $k \equiv 0$, prove that

$$\int_0^\ell w^2/k\,dx \int_0^\ell \bar{w}\bar{y}_0\ dx \geq \left[\int \bar{w}\ (x) y\,dx\right]^2$$

Hence, or otherwise, prove that

$$|y(a)| \leq \sqrt{\int_0^\ell w^2/k\,dx \int_0^a \frac{dx}{T(x)}}\,.$$

4. Let D be a plane domain with smooth boundary C, the boundary C consisting of two distinct portions C_1, C_2 such that $C = C_1 \cup C_2$ and $C_1 \cap C_2 = 0$. The functional

$$I = \int_D F(x,y,z,z_x,z_y)\,dA$$

is defined in the class of sufficiently smooth functions $z = z(x,y)$ which take an assigned value $z = f(x,y)$ on the portion of the boundary C_1. Obtain necessary conditions for a minimum of I in the form of a partial differential equation for $z = z(x,y)$ and a boundary condition

$$\frac{\partial F}{\partial z_x} v_x + \frac{\partial F}{\partial z_y} v_y = 0 \text{ on } C_2\,.$$

If

$$F = \sqrt{1 + z_x^2 + z_y^2}$$

prove that the Euler-Lagrange differential equation has the form

$$Az_{xx} + Cz_{xy} + Bz_{yy} = 0,$$

where A, B, C are functions of the first derivatives of z, which are to be deduced.