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GD1676

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2000

Second Engineering Examination

Mathematical Physics [MP250]

Paper Two

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Time allowed: *Three* hours

Answer THREE questions from Section A and TWO questions from Section B.

A list of useful information and mathematical formulae accompanies this examination paper.

SECTION A

1. The power input to an axial flow pump is thought to depend on the volume flowrate Q, the head H (energy per unit mass), the angular speed N and diameter D of the pump, and the density ρ of the fluid, *i.e.*

$$P = F(Q, H, N, D, \rho) .$$

Find the most general form of this functional dependence, making use of dimensional analysis.

An axial flow pump is required to deliver $30\,\mathrm{m^3\,s^{-1}}$ of water at a head of $120\,\mathrm{J\,kg^{-1}}$. The diameter of the rotor is 0.8 m and it is to be driven at 800 rpm. The prototype is to be modelled by a small test apparatus having a 16 W power supply and using water. The diameter of the model rotor is 0.08 m and it is to be driven at 2400 rpm.

For completely similar performance between the prototype and the model, calculate the head and flowrate for the model. Estimate, also, the power supplied to the prototype.

2. Two vertical wheels, each of mass m and radius a, are released simultaneously from rest at the top of an inclined plane and remain vertical while moving down the incline. The coefficient of friction is $\mu = \frac{5}{13}\tan\alpha$ where α is the angle of inclination of the plane. Wheel 1 is a uniform solid disk. Wheel 2 is also uniform but has a hole cut out of it (the hole centred at the centre of the wheel) so that its axial moment of inertia is $\frac{2}{3}ma^2$.

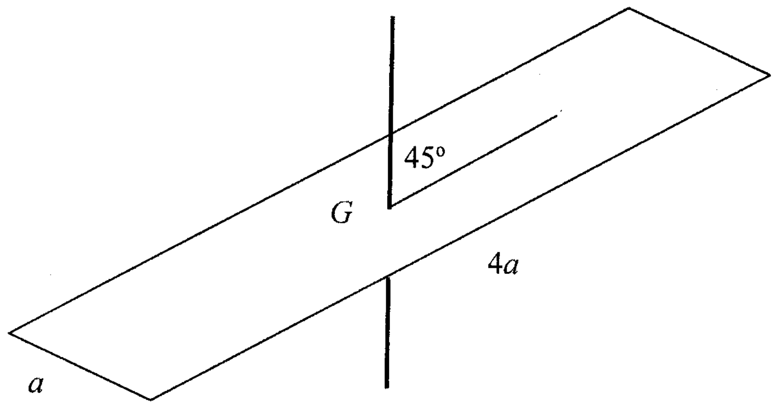
Show that wheel 1 will roll down the incline while wheel 2 will slide. Which wheel will get to the bottom first, assuming that they move along parallel straight paths down the incline? Justify your answer.

3. A cable whose uniform mass per unit length is $9\,\mathrm{kg/m}$ is suspended between two points on the same level $60\,\mathrm{m}$ apart. The sag of the cable is $5\,\mathrm{m}$.

   Find the maximum tension in the cable and the total length of the cable, given that $z = 0.33$ is the (approximate) non-trivial solution of the equation

$$z = 6(\cosh z - 1).$$

4. A uniform rectangular plate, with mass m and sides of length a and $4a$, spins with angular speed ω about a vertical axis which passes through its centre of mass G, as shown in the diagram below. The axis makes an angle of $45°$ with the line in the plate through G parallel to the longer side of the plate.
   Express the angular momentum of the plate relative to G in terms of the angular speed and the parameters of the plate.
   Compute the moment of inertia of the plate about the vertical axis of rotation.
   Determine the bending moment which the shaft exerts on the rotating plate and deduce, also, the reaction on the shaft due to the wobble of the plate.



5. A particle of mass $3m$ is free to slide on a fixed smooth straight horizontal wire. A second particle of mass $2m$ is suspended from the first particle by a heavy rigid rod, of length $2l$ and mass $4m$. The system is allowed to swing in the vertical plane of the rod and wire.
   Choosing as generalised coordinates the distance s of the mass $3m$ from a fixed point along the wire and the angle θ which the rod makes with the downward vertical, show that the Lagrangian function for this system is

$$\frac{9}{2} m\dot{s}^2 + 8ml\dot{s}\dot{\theta}\cos\theta + \frac{20}{3} ml^2\dot{\theta}^2 + 8mgl\cos\theta$$

   and, hence, obtain Lagrange's equations of motion for the system.

SECTION B

6. The curve C, described by the parametric equation

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 2t\,\mathbf{k}\,, \quad -\infty < t < \infty$$

passes through the point $\left(1, 2\sqrt{3}, \frac{2}{3}\pi\right)$. Find a unit tangent vector to the curve C at this point. Hence, or otherwise, find the derivative of the scalar field

$$\phi = 4xyz - 3y^2$$

along the curve at $\left(1, 2\sqrt{3}, \frac{2}{3}\pi\right)$.

Verify, by calculating each term separately, the identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \bullet \mathbf{A}) - \nabla^2\mathbf{A}$$

for the vector field $\mathbf{A} = y^2\,\mathbf{i} + x^2y\,\mathbf{j} + 2xz\,\mathbf{k}$.

7. Evaluate the line integral

$$\int_{(0,0,0)}^{(4,2,-3)} \mathbf{V} \bullet \mathbf{dr}$$

for the vector field $\mathbf{V} = 2yz\,\mathbf{i} - (x^2 + y^2)\,\mathbf{j} + 5xz\,\mathbf{k}$ along each of the following paths:
the successive straight line segments $(0,0,0)$ to $(0,2,0)$ to $(4,2,0)$ to $(4,2,-3)$,
the curved line $x = 4t^3,\ y = 2t^2,\ z = -3t$ from $t = 0$ to $t = 1$, and
the straight line joining $(0,0,0)$ directly to $(4,2,-3)$: $x = 2y,\ z = -\frac{3}{2}y$.

Can you deduce, based purely on the answers you obtained, whether the given vector field is conservative, or not? Justify your answer.

8. Verify the divergence theorem of Gauss for the vector field

$$\mathbf{V} = 2xz\,\mathbf{i} + yz\,\mathbf{j} + 3yz\,\mathbf{k}$$

when the volume of integration is the finite volume enclosed by the planes $x = 0, y = 0, z = 0,$ $x = 2, y = 4$ and $z = 3$.