OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

GX1673

SUMMER EXAMINATIONS, 2000

Second Science and Arts Examination
Mathematical Physics(Honours) MP203

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Time allowed: *Two* hours
At least THREE questions to be attempted,at least ONE from each section.

Section A

1. (a) Consider the integral

$$\int_0^1 \int_0^y x^2 e^{xy} dxdy$$

Sketch the region of integration. Change the order of integration and evaluate the integral

(b) Use plane polar coordinates to evaluate

$$\iint_A xydxdy$$

where A is the area whose boundary consists of the $x-axis$ between $x = 0$ and $x = 2$, the $y-axis$ between $y = 0$ and $y = 2$ and a quarter of the circle $x^2 + y^2 = 4$.

(c) Evaluate the integral

$$\iint_A xe^{x+y} dA$$

where A is the region bounded by the lines $x+y = 1,\ x+y = -1,\ x-y = 1,\ x-y = -1$ by making the change of variables $u = x-y,\ v = x+y$.

2. (a)Show that the point $(1,-2,1)$ lies on each of the surfaces

$$xy^2z = 3x+z^2 \text{ and } 3x^2-y^2+2z = 1$$

Find a unit normal to each surface at this point and deduce the acute angle between the surfaces at this point.

(b) Verify the identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \bullet \mathbf{B})\mathbf{A} - (\nabla \bullet \mathbf{A})\mathbf{B} + (\mathbf{B} \bullet \nabla)\mathbf{A} - (\mathbf{A} \bullet \nabla)\mathbf{B}$$

for the vector fields

$$\mathbf{A} = 2x\mathbf{i} + 3y\mathbf{j} + 5z\mathbf{k} \text{ and } \mathbf{B} = y\mathbf{i} - z\mathbf{j} + x\mathbf{k}.$$

3. (a)Evaluate the line integral

$$\int_{(0,0,0)}^{(4,2,-3)} \mathbf{V}_1 \cdot d\mathbf{r}$$

for the vector field $\mathbf{V}_1 = 2yz\mathbf{i} - (x^2 + y^2)\mathbf{j} + 5xz\mathbf{k}$ along each of the following paths:

the successive straight line segments from $(0,0,0)$ to $(0,2,0)$ to $(4,2,0)$ to $(4,2,-3)$

the curved line $x = 4t^3,\ y = 2t^2,\ z = -3t$ from $t = 0$ to $t = 1$

Can you deduce, based on your answers, whether the vector field $\mathbf{V}_1$ is conservative or not? Justify your answer.

(b)Show that the vector field

$$\mathbf{V}_2 = (22xz^2 - 3y)\mathbf{i} + (14yz - 3x)\mathbf{j} + (22x^2z + 7y^2)\mathbf{k}$$

is conservative. Find the corresponding scalar potential by evaluating the line integral

$$\int_{(0,0,0)}^{(x,y,z)} \mathbf{V}_2(x',y',z') \cdot d\mathbf{r}'$$

along a suitably chosen path.

Section B

4. The divergence theorem of Gauss can be written in the form

$$\iiint_V \nabla \cdot \mathbf{A} dV = \iint_S \mathbf{n} \cdot \mathbf{A} dS$$

Explain what V, S and $\mathbf{n}$ are in this equation.

Verify divergence theorem for the vector field

$$\mathbf{A} = xy\mathbf{i} + z^2\mathbf{j} - 2x^2z\mathbf{k}$$

for the cylindrical region $x^2 + y^2 \leq 9,\ -1 \leq z \leq 3$.

5. Suppose that an orthogonal curvilinear coordinate system (u_1, u_2, u_3) has unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and scale factors $\{h_1, h_2, h_3\}$.

a. If Φ is a scalar function of (u_1, u_2, u_3) then show that the gradient of Φ is given by

$$\nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{e}_3.$$

b. If $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ is a vector function of (u_1, u_2, u_3) then show that the divergence of $\mathbf{A}$ is given by

$$\nabla.\mathbf{A} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right).$$

c. Using the results of **a** and **b**, or otherwise, obtain an expression for the Laplacian of Φ in (u_1, u_2, u_3).

d. Calculate the expressions for the gradient and divergence in cylindrical polar coordinates.

6. **a.** Define cartesian tensors of ranks one and two.

b. Determine the rotation matrix associated with a rotation of a cartesian frame about its x_1 axis, the rotation being of angle α in an anti-clockwise direction. If the new frame is rotated by an angle β about the new x_3 axis, calculate the rotation matrix for the composite rotation.

c. A tensor of rank two has components

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

with respect to a given cartesian frame. The frame is rotated by an angle α about its x_1 axis (see part **b** of this question). Find the components of the tensor in the rotated frame.