GX1688

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2000

THIRD SCIENCE EXAMINATION
MP362
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Time allowed: *TWO* hours.
Full marks for *THREE* completed questions.

1. Consider the following initial boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L, t > 0,$$
$$u = 0 \text{ on } x = 0, t > 0,$$
$$\frac{\partial u}{\partial x} + \gamma u = 0 \text{ on } x = L, t > 0,$$
$$u = f(x) \text{ at } t = 0, 0 < x < L,$$

where $u = u(x,t)$ and α, L, γ are positive constants.

a. Letting $u(x,t) = X(x)T(t)$, show that
$$X'' + \lambda X = 0,\ X'(L) + \gamma X(L) = 0 \text{ and } T' + \alpha^2 \lambda T = 0$$
where λ is the separation constant.

b. For $\lambda > 0$, show that eigenvalues are the solutions for λ to $\tan(\sqrt{\lambda} L) = -\sqrt{\lambda}/\gamma$. Demonstrate graphically that this equation has an infinity of solutions for λ.

c. Construct the eigenfunctions and hence obtain the solution for u. Investigate the limits $\gamma \to 0^+$ and $\gamma \to \infty$.

2. Consider the following two point boundary value problem:
$$L(y) = \lambda r(x) y,\ 0 < x < 1,$$
$$a_1 y(0) + a_2 y'(0) = 0,\ b_1 y(1) + b_2 y'(1) = 0$$
where
$$L(y) = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x) y$$
and where $p(x), q(x), r(x), a_1, a_2, b_1, b_2$ satisfy all the usual conditions for the above problem to be regular.

a. If $u(x), v(x)$ are any two twice continuously differentiable real valued functions, then show that
$$(L(u), v) - (u, L(v)) = -p(x)(u'(x)v(x) - u(x)v'(x))\big|_0^1$$
where $(u,v) = \int_0^1 u(x)v(x)dx$ is the usual inner product. If, in addition, $u(x), v(x)$ satisfy the boundary conditions on $x = 0, 1$ given above, then show that
$$(L(u), v) = (u, L(v)).$$

b. Determine the normalised eigenfunctions of
$$y'' + \lambda y = 0,\ y(0) = y'(1) = 0.$$
Find the eigenfunction expansion of $f(x) = x$, $0 < x < 1$ in terms of these eigenfunctions.

3. Consider the following initial boundary value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right),\ 0 \le r < 1, t > 0,$$
$$u = 0 \text{ on } r = 1, t > 0,$$
$$u = f(r),\ \frac{\partial u}{\partial t} = 0 \text{ at } t = 0,\ 0 \le r < 1$$

where $u = u(r,t)$, a is a constant and $f(r)$ is a given smooth function.

a. Letting $u(r,t) = R(r)T(t)$, show that
$$r^2R'' + rR' + \lambda r^2 R = 0,\ R(1) = 0 \text{ and } T'' + a^2\lambda T = 0$$
where λ is the separation constant.

b. Show that the eigenvalues are the solutions for λ to $J_0(\sqrt{\lambda}) = 0$ where J_0 is a Bessel function of order zero of the first kind.

c. Construct the eigenfunctions and hence obtain the solution for u.

d. Calculate the solution for u if the initial conditions are changed to
$$u = 0,\ \frac{\partial u}{\partial t} = g(r) \text{ at } t = 0,\ 0 \le r < 1$$
where $g(r)$ is another given smooth function.
Note. The equation $r^2R'' + rR' + \lambda r^2R = 0$ has the general solution $R(r) = AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r)$ where A, B are arbitrary constants and J_0, Y_0 are Bessel functions of order zero of the first and second kind, respectively.

4. Consider the following initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0,$$
$$u = f(x) \text{ at } t = 0, -\infty < x < \infty$$

where $u = u(x,t)$, K is a positive constant and $f(x)$ is a given smooth function. The intervals $-\infty < x < \infty$ and $t > 0$ are split into the uniform grids
$$\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots \text{ and } 0 = t_0 < t_1 < t_2 < \ldots$$
where $x_j = j(\Delta x), j = \ldots,-2,-1,0,1,2,\ldots$ and $t_k = k(\Delta t), k = 0,1,2,\ldots$ with a view to solving the problem numerically using finite difference techniques. The numerical approximation to u at gridpoint (x_j, t_k) is denoted by U_j^k.

a. Using standard finite difference approximations for the derivatives, derive the following implicit time-stepping scheme for solving the above problem:
$$-KhU_{j-1}^k + (1+2Kh)U_j^k - KhU_{j+1}^k = U_j^{k-1}$$
where $h = \Delta t/(\Delta x)^2$.

b. Denote by e_j^k the error in the implementation of the above difference scheme; these errors also satisfy the scheme. By considering a representative fourier term $e_j^k = \exp(i(\theta j + \lambda k))$ $(i^2 = -1)$ where θ and λ are constants, show that
$$\exp(-i\lambda) = 1 + 4Kh\sin^2(\theta/2)$$
and hence show that the implicit scheme is von Neumann stable for all $h > 0$.

c. Using central difference approximations for the time derivative gives the following scheme for the heat equation:
$$U_j^{k+1} - U_j^{k-1} = Kh(U_{j+1}^k - 2U_j^k + U_{j-1}^k).$$
Show that this scheme is von Neumann unstable for all $h > 0$.

5. a. Let $f(x)$ be a real valued function defined on $-\infty < x < \infty$ which has fourier transform $\overline{f(c)}$. Show that (Parseval's equation)

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\overline{f(c)}|^2 dc.$$

b. Consider the following initial boundary value problem for the heat equation on a semi-infinite domain:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < \infty, t > 0,$$
$$u = 0 \text{ on } x = 0, t > 0,$$
$$u = f(x) \text{ at } t = 0, 0 < x < \infty$$

where $u = u(x,t)$ and $f(x)$ is a given smooth function. You may assume that u and its deivatives tend to zero as $x \to \infty$. Solve this problem by taking fourier sine transforms in the x variable.

Notes: (i) The fourier transform and fourier inverse transform of a function $f(x)$, $-\infty < x < \infty$, are

$$\overline{f(c)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{icx} f(x) dx \text{ and } f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-icx} \overline{f(c)} dc.$$

(ii) The fourier sine transform and fourier inverse sine transform of a function $f(x)$, $0 < x < \infty$, are

$$\overline{f_s(c)} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin cx dx \text{ and } f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \overline{f_s(c)} \sin cx dc.$$