GXL690

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2000

THIRD SCIENCE EXAMINATION
MP323 ELECTROMAGNETISM

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Time allowed: *TWO* hours.
Attempt *THREE* questions.

1. **a.** A certain distribution of electric charge is spherically symmetric about the origin and the total charge inside a sphere of radius r (centre the origin) is $Q(r)$. Show that the electrostatic potential $V(r)$ is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_r^\infty \frac{Q(t)}{t^2}dt.$$

Show that this may be written in the equivalent form

$$V(r) = \frac{Q(r)}{4\pi\varepsilon_0 r} + \frac{1}{\varepsilon_0}\int_r^\infty s\rho(s)ds$$

where $\rho(r)$ is the charge density at distance r from the origin.

b. Show that the potential at the centre of a square sheet uniformly charged with charge density σ is given by

$$V_0 = \frac{\sigma d}{\pi\varepsilon_0}\ln(1+\sqrt{2})$$

where d is the length of a side of the square.

2. A current distribution $\mathbf{J}(\mathbf{r}')$ is localised in a finite region of space $V(\mathbf{r}')$ containing the origin O. The magnetic vector potential $\mathbf{A}(\mathbf{r})$ due to this current distribution is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV(\mathbf{r}')$$

where μ_0 is the permeability of free space.

a. By expanding $1/|\mathbf{r}-\mathbf{r}'|$ in an appropriate manner, show that for $|\mathbf{r}| \gg |\mathbf{r}'|$,

$$A_i(\mathbf{r}) \approx \frac{\mu_0}{4\pi|\mathbf{r}|}\int J_i(\mathbf{r}')dV(\mathbf{r}') + \frac{\mu_0}{4\pi}\frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\int \mathbf{r}'J_i(\mathbf{r}')dV(\mathbf{r}')$$

where $\mathbf{A} = (A_i)$ and $\mathbf{J} = (J_i)$, $i = 1,2,3$.

b. Show that

$$\int J_i(\mathbf{r}')dV(\mathbf{r}') = 0 \text{ and } \int(\mathbf{r}.\mathbf{r}')J_i(\mathbf{r}')dV(\mathbf{r}') = -\frac{1}{2}\int \mathbf{r}\times(\mathbf{r}'\times\mathbf{J}(\mathbf{r}'))dV(\mathbf{r}').$$

Hence show that for $|\mathbf{r}| \gg |\mathbf{r}'|$,

$$\mathbf{A} \approx \frac{\mathbf{m}\times\mathbf{r}}{|\mathbf{r}|^3} \text{ where } \mathbf{m} = \frac{\mu_0}{8\pi}\int \mathbf{r}'\times\mathbf{J}(\mathbf{r}')dV(\mathbf{r}').$$

Note. You may quote the vector identity $(\mathbf{r}.\mathbf{r}')\mathbf{J} = (\mathbf{r}.\mathbf{J})\mathbf{r}' - \mathbf{r}\times(\mathbf{r}'\times\mathbf{J})$.

c. Use the results of part **b** of this question to show that for $|\mathbf{r}| \gg |\mathbf{r}'|$,

$$\mathbf{B}(\mathbf{r}) \approx \frac{3\hat{\mathbf{r}}(\mathbf{m}.\hat{\mathbf{r}}) - \mathbf{m}}{|\mathbf{r}|^3}$$

where $\mathbf{B}(\mathbf{r})$ is the magnetic field and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.

p.t.o.

3. A solid sphere of radius a is made of dielectric material which has dielectric constant ε. The sphere is placed in $0 \leq r < a$ in an initially uniform electric field of intensity $\mathbf{E} = E_o\hat{\mathbf{k}}$. Using standard notation, the electrostatic potential is written as

$$\Phi = \Phi_1(r,\theta) \text{ for } r < a \text{ and } \Phi = \Phi_2(r,\theta) \text{ for } r > a.$$

 a. Explain why the appropriate boundary conditions are

$$\lim_{r\to a^-}\left(\varepsilon\frac{\partial\Phi_1}{\partial r}\right) = \lim_{r\to a^+}\left(\varepsilon_0\frac{\partial\Phi_2}{\partial r}\right) \text{ and } \lim_{r\to a^-}\left(\frac{\partial\Phi_1}{\partial\theta}\right) = \lim_{r\to a^+}\left(\frac{\partial\Phi_2}{\partial\theta}\right)$$

 where ε_0 is the permittivity of free space.

 b. Calculate the potentials Φ_1 and Φ_2.

 c. Calculate the charge density of polarization on the surface of the sphere.

4. Maxwell's equations for a linear non-conducting medium are, in the usual notation,

$$\nabla.\mathbf{E} = 0,\ \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0},\ \nabla.\mathbf{B} = 0,\ \nabla\times\mathbf{B} - \mu\varepsilon\frac{\partial\mathbf{E}}{\partial t} = \mathbf{0}.$$

 a. Show that both $\mathbf{E}$ and $\mathbf{B}$ satisfy the wave equation with wave speed $c = 1/\sqrt{\mu\varepsilon}$.

 b. Consider plane wave solutions of the above equations for $\mathbf{E}$ and $\mathbf{B}$ of the form

$$\mathbf{E}(\mathbf{r},t) = \boldsymbol{\varepsilon}_1 E_0\exp(i(\mathbf{k}.\mathbf{r} - \omega t)),\ \mathbf{B}(\mathbf{r},t) = \boldsymbol{\varepsilon}_2 B_0\exp(i(\mathbf{k}.\mathbf{r} - \omega t))$$

 where $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$ are constant unit vectors and E_0, B_0 are constant complex amplitudes. The wave vector $\mathbf{k}$ and the frequency ω are also constant. Show that

$$\boldsymbol{\varepsilon}_1.\mathbf{k} = \boldsymbol{\varepsilon}_2.\mathbf{k} = 0, \boldsymbol{\varepsilon}_2 = \frac{\mathbf{k}\times\boldsymbol{\varepsilon}_1}{|\mathbf{k}|}, B_0 = \sqrt{\mu\varepsilon}\,E_0.$$

 c. Calculate the average of the energy flux $\mathbf{S}$ and the energy density u over one period of the wave. Hence show that $\langle\mathbf{S}\rangle = c\langle u\rangle\boldsymbol{\varepsilon}_3$ where $\langle\ \ \rangle$ denotes the averaged quantities and $\boldsymbol{\varepsilon}_3 = \boldsymbol{\varepsilon}_1\times\boldsymbol{\varepsilon}_2$. You may assume that E_0 and B_0 are real.

 Note. The energy flux $\mathbf{S}$ and energy density u are given by, in the usual notation,

$$\mathbf{S} = \mathbf{E}\times\mathbf{H},\ u = \frac{1}{2}(\mathbf{E}.\mathbf{D} + \mathbf{B}.\mathbf{H}).$$

5. The electric vector in a circularly polarised plane electromagnetic wave in vacuo has components

$$E_x = \alpha\cos\beta(t + z/c_0),\ E_y = \alpha\sin\beta(t + z/c_0),\ E_z = 0$$

where $c_0 = 1/\sqrt{\mu_0\varepsilon_0}$ is the speed of light in vacuo and α, β are constants. Determine the magnetic vector.

The wave is incident normally on the plane face of a mass of uniform dielectric $\mu = 1, \varepsilon$, occupying the region $z < 0$. Show that the reflected wave is also circularly polarised and find the electric and magnetic vectors in the reflected wave.

Notes. (i) The incident wave may be written as $\mathbf{E}^i = \mathbf{E}_0^i\exp(-i\beta(t + z/c_0))$ where $\mathbf{E}_0^i = \alpha(1, i, 0)$ and we interpret the real part as the physical wave.

(ii) If a surface S with normal $\mathbf{n}$ separates two dielectric media, then the boundary conditions across this surface are, in the usual notation,

$$(\mathbf{B}' - \mathbf{B}).\mathbf{n} = 0,\ (\mathbf{H}' - \mathbf{H})\times\mathbf{n} = \mathbf{0},\ (\mathbf{D}' - \mathbf{D}).\mathbf{n} = 0,\ (\mathbf{E}' - \mathbf{E})\times\mathbf{n} = \mathbf{0}$$

where the primed quantities denote the fields on one side of the surface and the unprimed quantities denote the fields on the other side. It is assumed that there is no surface charge or current.